On the angular momentum density of a two-dimensional quantum Heisenberg antiferromagnet

1998 J. Phys. A: Math. Gen. 31 5401

(http://iopscience.iop.org/0305-4470/31/23/020)

J. Phys. A: Math. Gen. **31** (1998) 5401–5411. Printed in the UK PII: S0305-4470(98)92149-7

On the angular momentum density of a two-dimensional quantum Heisenberg antiferromagnet

Simon Villain-Guillot† and Rossen Dandoloff‡

† Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany

‡ Laboratoire de Physique Théorique et Modélisation, Université de Cergy-Pontoise, BP 8428, 95806 Cergy-Pontoise Cedex, France

Received 25 February 1998

Abstract. We study Heisenberg spins on an infinite plane. In the continuum limit the Hamiltonian of the system is given by the nonlinear σ model. Following an approach developed by Mikeska and Affleck, we find that the angular momentum associated with the order parameter presents a classical spin part, associated with the gauge freedom of a trihedra. We show that this gauge field may induce a non-trivial topological term, the Hopf term (or Chern–Simons term), as initially suggested by Dzyaloshinski, Polyakov and Wiegmann.

1. Introduction

We study the continuum limit of a system of Heisenberg spins on a square lattice in the infinite plane. The Hamiltonian of the system is given by the nonlinear σ model. In one dimension, it is now accepted that, in the Lagrangian, there is an additional topological term, the Pontryagin index. This was initially suggested by Haldane [1]. It is constructed from the topological current density $J^1 = \boldsymbol{n} \cdot \partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n}$.

In dimension $D = 2+1$, there are two procedures to construct a topological term. First, we can generalize J^1 to a three-dimensional current, $J^i = \varepsilon^{ijk} \boldsymbol{n} \cdot \partial_j \boldsymbol{n} \times \partial_k \boldsymbol{n}$, from which one can derive the following three-dimensional topological term

$$\int \operatorname{div} \boldsymbol{J} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{1}$$

However, $\boldsymbol{J}$ is a closed two-form, therefore $\operatorname{div} \boldsymbol{J}$ vanishes identically, and the above integral gives no contribution [2]. In differential geometry, this is the Bianchi equation [3]. Owing to this property, one can define a gauge field $\boldsymbol{A}$ such that rot $\boldsymbol{A} = \boldsymbol{J}$. It is this gauge field which enables us to build yet another different topological term, the Hopf term [4]

$$H_{\mathrm{Hopf}} = \int \boldsymbol{A} \cdot \boldsymbol{J} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{2}$$

As initially suggested by Dzyaloshinski *et al* [5], this is the proper term in dimension $D = 2 + 1$ which should appear in the Lagrangian of the antiferromagnet in the continuum limit. It indexes topologically the different mapping of S^3 to S^2 (where S^3 results from the compactification of $2 + 1$).

Previous works [6–8], while inquiring into the possible extension in dimension $2 + 1$ of the Haldane result, that is looking for an effective continuous action starting from the microscopic Heisenberg Hamiltonian, have focused on the first term (1), when computing the local anaholonomy. In this paper, we follow the same approach as Mikeska [9] and Affleck [10]. In addition to the order parameter and generator of rotation, we introduce two auxiliary fields which can be interpreted as covariant derivatives of the order parameter and which enable us to introduce a gauge field. In the Hamiltonian expansion, we shall keep the contributions involving this gauge field, i.e. we shall keep possible non-local terms, and show that they lead to a non-trivial topological term in the Lagrangian. Moreover, while taking the continuum limit, we shall compute the Heisenberg interaction terms, in such a way that we shall remove irrelevant contributions that could occur from the breaking of the translational symmetry characteristic of the Affleck procedure [11].

2. Definition of a continuous field theory in the one-dimensional case

Consider a one-dimensional Heisenberg antiferromagnet chain of spins. In the case of a nearest-neighbour interaction, the Hamiltonian is given by

$$H_{\mathrm{Heisenberg}} = J \sum_{\langle i,j\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{3}$$

where J is the positive coupling constant. The local degree of freedom is represented by an operator $\boldsymbol{S}$, satisfying the Poisson brackets

$$\{S_i^a, S_j^b\} = \mathrm{i}\varepsilon^{abc} S_j^c \delta_{ij}. \tag{4}$$

In the classical limit (large S) and when looking for low-energy excitations, we can suppose that the system has a staggered magnetization: the chain is locally almost Néel ordered. Under this assumption, $\boldsymbol{S}_{2i} + \boldsymbol{S}_{2i+1}$ will be small (of the order of a, the lattice spacing). Following Mikeska and Affleck, we introduce an elementary cell, of size $2a$ (it now includes two spins) and decompose the variable $\boldsymbol{S}$ into slow and fast varying components. These two fields are defined in the following way

$$\begin{aligned} \boldsymbol{n}(x) &= \frac{1}{2s}(\boldsymbol{S}_{2i} - \boldsymbol{S}_{2i+1}) \\ \boldsymbol{l}(x) &= \frac{1}{2a}(\boldsymbol{S}_{2i} + \boldsymbol{S}_{2i+1}). \end{aligned} \tag{5}$$

They correspond to the Fourier components of $\boldsymbol{S}$ for the $q = 0$ and $q = \pi$ modes. These are the two modes which are relevant when studying the long-distance (low-energy) behaviour [1]. These two components satisfy the following relation

$$\boldsymbol{n}^2 = 1 - \frac{a^2}{s^2}\boldsymbol{l}^2 \qquad \text{and} \qquad \boldsymbol{n}\cdot\boldsymbol{l} = 0. \tag{6}$$

In the continuum limit ($a \to 0$), $\boldsymbol{n}^2 = 1$, i.e. $\boldsymbol{n}$ lives on the sphere S^2. The second relation (which is exact in $D = 1 + 1$) tells us that the two fields $\boldsymbol{l}$ and $\boldsymbol{n}$ are orthogonal. Thus, we find that there are $6 - 2 = 4$ degrees of freedom per elementary cells of length $2a$, as expected from the initial microscopic Hamiltonian.

In the continuum limit $\lim_{a\to 0} \frac{\delta_{x_1,x_2}}{2a} = \delta(x_1 - x_2)$; therefore, these two fields generate simple Poisson Brackets:

$$\begin{aligned} \{n^i(x_1), n^j(x_2)\} &= \mathrm{i}\frac{4a^2}{s^2}\varepsilon^{ijk} l^k \delta(x_1 - x_2) \to 0 \\ \{l^i(x_1), l^j(x_2)\} &= \mathrm{i}\varepsilon^{ijk} l^k \delta(x_1 - x_2) \\ \{l^i(x_1), n^j(x_2)\} &= \mathrm{i}\varepsilon^{ijk} n^k \delta(x_1 - x_2). \end{aligned} \tag{7}$$

Thus we conclude that the order parameter $\boldsymbol{n}$ is a free field of tridimensional vectors, of length one. Moreover, the SO(3) Lie algebra structure generated by the Poisson Brackets for $\boldsymbol{l}$ indicates that this field is the generator of rotation for $\boldsymbol{n}$.

The Liouville theorem allows us to establish another relation between $\boldsymbol{l}$ and $\boldsymbol{n}$. We apply it to $\boldsymbol{S}_{2i}$ and $\boldsymbol{S}_{2i+1}$ successively, and, when expanding the order parameter $\boldsymbol{n}$ in the neighbourhood of x, the centre of the elementary cell, we can write:

$$\boldsymbol{l} = \boldsymbol{n} \times \boldsymbol{\Pi} = \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{cg^2 \partial t} - \frac{\alpha}{4\pi} \partial_x \boldsymbol{n}(x). \tag{8}$$

Here $\boldsymbol{\Pi}$ is the momentum conjugate to $\boldsymbol{n}$. We have also defined $c = 2Jas$, the velocity of the magnons for a unidimensional antiferromagnet together with $g^2 = \frac{2}{s}$ and $\alpha = 2\pi s$.

When expanding the Heisenberg Hamiltonian (3) as a function of the variables $\boldsymbol{l}$, $\boldsymbol{n}$ and $\frac{\partial \boldsymbol{n}}{\partial x}$, we find the following expression:

$$H_{\text{Heisenberg}} = ac \sum_{\langle i,j \rangle} g^2 \left(\boldsymbol{l}^2 + s\boldsymbol{l}(x) \cdot \partial_x \boldsymbol{n}(x) + \frac{s^2}{4} (\partial_x \boldsymbol{n}(x))^2 \right) + \frac{1}{g^2} (\partial_x \boldsymbol{n}(x))^2. \tag{9}$$

In the limit where a goes to 0, $2a \sum_{\langle i,j \rangle} \longrightarrow \int \mathrm{d}x$. Therefore, we can write

$$H_{\text{Heisenberg}} = \frac{c}{2} \int g^2 \left(\boldsymbol{l} + \frac{\alpha}{4\pi} \partial_x \boldsymbol{n} \right)^2 + \frac{1}{g^2} (\partial_x \boldsymbol{n})^2 \, \mathrm{d}x \tag{10}$$

where $\alpha = 2\pi s$. If we replace $\boldsymbol{l}$ with the expression (8), we find

$$H_{\text{Heisenberg}} = \frac{1}{2} \int \frac{1}{cg^2} (\partial_t \boldsymbol{n})^2 + \frac{c}{g^2} (\partial_x \boldsymbol{n})^2 \, \mathrm{d}x. \tag{11}$$

Thus, the long-distance behaviour of the infinite one-dimensional chain of Heisenberg spins is given by the nonlinear σ model. A Legendre transformation, using the conjugate momenta given by relation (8), leads, for this system, to the following Lagrangian

$$L = \frac{1}{2g^2} \iint \mathrm{d}x \, \mathrm{d}t \left[\frac{1}{c} \left(\frac{\partial \boldsymbol{n}}{\partial t} \right) - c (\partial_x \boldsymbol{n})^2 \right] + c \frac{\alpha}{4\pi} \boldsymbol{n} \cdot \partial_x \boldsymbol{n} \times \partial_t \boldsymbol{n}. \tag{12}$$

This is the Lagrangian of the nonlinear σ model plus a total derivative corresponding to a topological term: *the Pontryagin index*. As $\boldsymbol{n}^2 = 1$, we can express it as a function of the coordinates of the sphere (θ, φ). The topological term then writes:

$$L_{1+1} = \frac{\alpha}{4\pi} \sin\theta (\partial_x \theta \partial_t \varphi - \partial_t \theta \partial_x \varphi). \tag{13}$$

It represents the Jacobian of the coordinates transformation $(\theta, \varphi) \rightarrow (x, y)$. Because $\alpha = 2\pi s$, the Pontryagin index is relevant only for half-integer spin chains. Haldane was the first to suggest that this kind of term could explain why integer and half-integer spin chains have different behaviours. The renormalization group analysis, which predict for the nonlinear σ model a gap in the excitation spectrum, is no longer relevant for the half-integer spin chains. As in 't Hooft's approach [12], the gap closes for $\alpha = \pi$, in agreement with the Lieb–Schultz–Mattis theorem [13].

3. Conservation of the number of degrees of freedom in the two-dimensional case

During our procedure of shrinking the lattice spacing of the chain to zero, we have kept constant the physical observables c and g^2, the velocity of the magnons and the coupling constant. We are now going to consider spins on a two-dimensional square lattice, and shall recover, using the Affleck procedure, the nonlinear σ model as the continuum limit of the

Hamiltonian of the system. However, in $D = 2 + 1$, the conjugate momenta Π_θ and Π_φ will have different expressions. Moreover, we shall see that a third conjugate momentum is required in order to fully describe the microscopic configuration.

The summation is now on each couple (i, j) of spins on next-neighbouring sites of a square lattice, where each link is counted only once. For this bipartite lattice, we have again, as in the one-dimensional chain, two square ferromagnetic sublattices, and we shall consider that the ground state is again an alternate state. Once more, we shall decompose the original order parameter into a slow varying mode, $\boldsymbol{n}$, and a local magnetization $\boldsymbol{l}$, associated with a rapidly varying mode, for a *plaquette* of size $2a \times 2a$.

Note that now, in order to conserve the number of degrees of freedom, we must also define, in addition to $\boldsymbol{n}$ and $\boldsymbol{l}$, two auxiliary fields $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$. These four fields can be expressed as functions of the original degrees of freedom as follows

$$\begin{aligned}
\boldsymbol{n} &= \frac{\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{d}}{4s} & \frac{\boldsymbol{a}}{s} &= \boldsymbol{n} + a^2\frac{\boldsymbol{l}}{s} + a(\boldsymbol{D}_x + \boldsymbol{D}_y)\\
\boldsymbol{l} &= \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}{4a^2} & \frac{\boldsymbol{b}}{s} &= a^2\frac{\boldsymbol{l}}{s} - \boldsymbol{n} + a(\boldsymbol{D}_y - \boldsymbol{D}_x)\\
\boldsymbol{D}_x &= \frac{\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c}+\boldsymbol{d}}{4as} & \frac{\boldsymbol{c}}{s} &= \boldsymbol{n} + a^2\frac{\boldsymbol{l}}{s} - a(\boldsymbol{D}_x + \boldsymbol{D}_y)\\
\boldsymbol{D}_y &= \frac{\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\boldsymbol{d}}{4as} & \frac{\boldsymbol{d}}{s} &= a^2\frac{\boldsymbol{l}}{s} - \boldsymbol{n} + a(\boldsymbol{D}_x - \boldsymbol{D}_y).
\end{aligned}$$

They correspond to the Fourier modes of the spin operators, with momenta near (π, π), $(0, 0)$, $(\pi, 0)$, $(0, \pi)$, which correspond to the dominant contributions in the low excitations regime. We have the following constraint

$$\boldsymbol{n}^2 = 1 - \frac{a^4}{s^2}\boldsymbol{l}^2 - a^2\boldsymbol{D}_x^2 - a^2\boldsymbol{D}_y^2. \tag{14}$$

In the continuum limit, the Poisson brackets for $\boldsymbol{n}$ vanish. Therefore in this limit, we again find that $\boldsymbol{n}$ is a free field and that the order parameter manifold is the sphere S^2. Note that in the $D = 2 + 1$ case, $\boldsymbol{n} \cdot \boldsymbol{l} \neq 0$.

The Poisson brackets involving $\boldsymbol{l}$ still induce a non-trivial algebra. Indeed, we can write:

$$\begin{aligned}
\{l^i(x_1, y_1), l^j(x_2, y_2)\} &= \mathrm{i}\varepsilon^{ijk} l^k \delta(x_1 - x_2)\delta(y_1 - y_2)\\
\{l^i(x_1, y_1), t^j(x_2, y_2)\} &= \mathrm{i}\varepsilon^{ijk} t^k \delta(x_1 - x_2)\delta(y_1 - y_2)
\end{aligned}$$

where $\boldsymbol{t}$ stands for any of the fields $\boldsymbol{n}$, $\boldsymbol{D}_x$ or $\boldsymbol{D}_y$. Thus, we can conclude that $\boldsymbol{l}$ is again the generator of rotations in the space where not only the order parameter $\boldsymbol{n}$ but also the two auxiliary fields $\boldsymbol{D}_i$ are defined. These last two fields satisfy the following Poisson bracket:

$$\{D_x^i(x_1, y_1), D_y^j(x_2, , y_2)\} = \frac{\mathrm{i}}{s}\varepsilon^{ijk} n^k \delta(x_1 - x_2)\delta(y_1 - y_2). \tag{15}$$

In contrast to [7], in the present derivation, we take into account these auxiliary fields. Thus we will recover the expected values for the physical observable c and g^2 [14].

4. The auxiliary fields

If we consider that $\frac{a-b}{2s}$ is a first-order approximation for $\boldsymbol{n}$, then we can consider that $2\boldsymbol{D}_x$ is almost the first derivative, along the direction of the axis $\boldsymbol{e}_x$, of the order parameter $\boldsymbol{n}$. However, the true definition of the derivative would require the use of the values of the

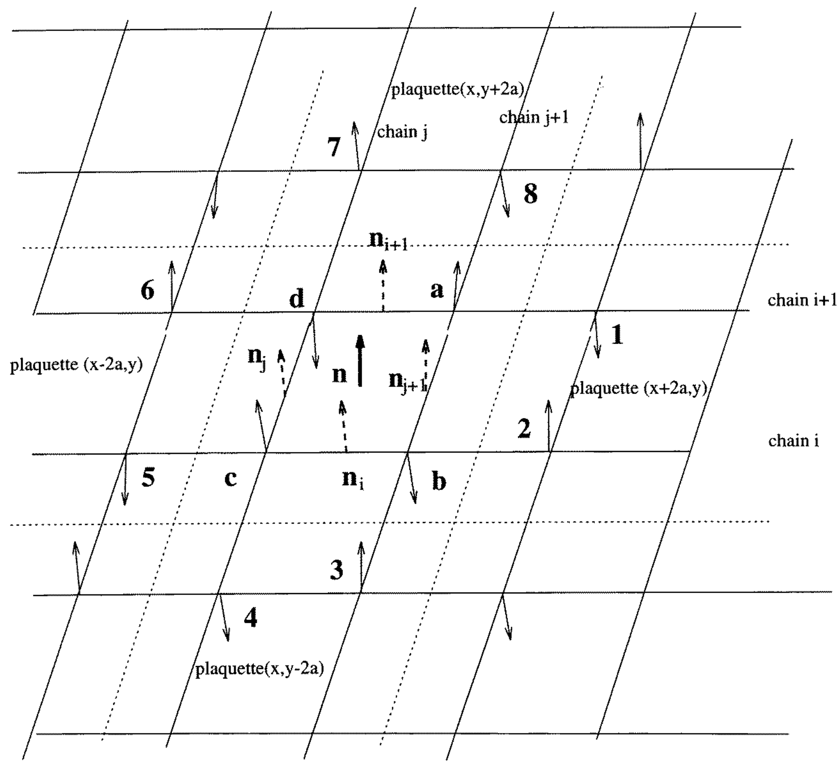

Figure 1. Two-dimensional Heisenberg antiferromagnet and the Affleck order parameter. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are the four spins on the *plaquette* (x, y). **1** and **2** belong to the *plaquette* $(x + 2a, y)$. $\boldsymbol{n}_j = \boldsymbol{c} - \boldsymbol{d}$ and $\boldsymbol{n}_{j+1} = \boldsymbol{a} - \boldsymbol{b}$ are the two pseudo-order parameters which we have used to define the covariant derivative $\boldsymbol{D}_x$.

field $\boldsymbol{n}$ defined on the neighbouring *plaquettes*. And, as $\boldsymbol{n}^2 = 1$, it would result in a field orthogonal to $\boldsymbol{n}$. However, here we have $\boldsymbol{D}_x \cdot \boldsymbol{n} \sim \boldsymbol{l} \cdot \boldsymbol{D}_y \neq 0$.

$\boldsymbol{D}_x$ is therefore a covariant derivative, which involves the gauge field associated with the parallel transport from one *plaquette* to another.

The same argument can be applied to characterize the field $\boldsymbol{D}_y$: we can associate it with the covariant derivative, along the direction of the axis $\boldsymbol{e}_y$.

$$\lim_{a\to 0} 2\boldsymbol{D}_\mu(x, y) = \partial_\mu \boldsymbol{n} + \frac{1}{s}\boldsymbol{\Gamma}_\mu$$

where $\boldsymbol{\Gamma}_\mu = A_\mu \boldsymbol{n}$ are connections. They express the deviation between our local pseudodefinition of the derivative and the true one which should be constructed along a geodesic in the order parameter manifold.

They are such that the auxiliary fields satisfy the Poisson brackets (15). As this relation can be interpreted as the Poisson brackets between covariant derivatives, it gives us the field strength tensor, or curvature, which is determined by $\boldsymbol{n}(x, y)$. The natural continuous gauge field in our problem is defined on site i by [15]:

$$A_\mu = \eta_i \boldsymbol{A}(\boldsymbol{n}_i) \cdot \partial_\mu \boldsymbol{n}_i = \eta_i \tau^g_\mu(i) \tag{16}$$

where $\eta_i = \pm 1$ for each of the sublattices. $\boldsymbol{A}$ is the monopole vector potential introduced by Haldane, and it satisfies: $\mathrm{rot}_{\boldsymbol{n}} \boldsymbol{A} = \boldsymbol{n}$. τ^g can be associated with the geodesic torsion of a space curve whose tangent is given by $\boldsymbol{n}$ [16]. We do not fix the gauge. In terms of Euler angles, the torsion is given by the following expression:

$$\tau_\mu^g = \boldsymbol{A}(\boldsymbol{n}) \cdot \partial_\mu \boldsymbol{n} = \cos(\theta)\partial_\mu \varphi + \partial_\mu \psi. \tag{17}$$

This gauge field expresses the parallel transport laws of the pseudo-order parameter $\frac{a-b}{2s}$. Note that in contrast to the $D = 1 + 1$ case, the true order parameter is now a full trihedra $(\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{v})$, where the gauge freedom ψ refers to the rotation in the $(\boldsymbol{u}, \boldsymbol{v})$ plane. This rotation enables us to connect locally the trihedras defined on two neighbouring *plaquettes* and allows for non-trivial mapping from S^3 to S^2. If the spin configuration in the compactified $D = 2 + 1$ space is trivial, then ψ can be removed by a global gauge transformation. However, this is no longer possible when the spin configuration involves anomalies, that is when the Hopf term is non-vanishing [17].

In order to check these expressions for the auxiliary fields as a function of $\boldsymbol{n}$, we can consider the connections $\boldsymbol{D}_\mu \cdot \boldsymbol{n}$ as the phase associated with $\boldsymbol{D}_\mu$ in the coherent state representation. Indeed, this representation enables us to express the scalar product between neighbouring spins [15]. For antiferromagnetically correlated configurations, we have

$$\langle \boldsymbol{\Omega}_i | \boldsymbol{\Omega}_{i+\delta\mu} \rangle = \left(\frac{1 - \boldsymbol{\Omega}_i \cdot \boldsymbol{\Omega}_{i+\delta\mu}}{2} \right)^s \mathrm{e}^{-\mathrm{i}A^\mu(\boldsymbol{x}_i)|\delta\mu|}. \tag{18}$$

In order to compute $\boldsymbol{D}_\mu \cdot \boldsymbol{n}$, we parallel transport each vector of the *plaquette* in the same manner, for example, along a path from $\boldsymbol{x}_i$ to $\boldsymbol{x}_f = \boldsymbol{x}_i + \boldsymbol{e}_x + \boldsymbol{e}_y$ with a right turn. Here $|\boldsymbol{e}_\mu| = \frac{a}{2}$. We take $\eta = +1$ for the sublattice which contains $\boldsymbol{a}$ and $\boldsymbol{c}$, that is when we go from a spin up to a spin down, and -1 if we go from a spin down to a spin up. We then find

$$\langle \boldsymbol{n} | \boldsymbol{a} \rangle = \mathrm{e}^{-\mathrm{i}\frac{a}{2}(\tau_y^g(\boldsymbol{x}_i) - \tau_x^g(\boldsymbol{x}_i + \boldsymbol{e}_y))} \qquad \langle \boldsymbol{n} | \boldsymbol{c} \rangle = \mathrm{e}^{-\mathrm{i}\frac{a}{2}(-\tau_y^g(\boldsymbol{x}_i) + \tau_x^g(\boldsymbol{x}_i - \boldsymbol{e}_y))}. \tag{19}$$

The two other terms give

$$-\langle -\boldsymbol{n} | \boldsymbol{b} \rangle = -\mathrm{e}^{\mathrm{i}\frac{a}{2}(\tau_x^g(\boldsymbol{x}_i) + \tau_y^g(\boldsymbol{x}_i + \boldsymbol{e}_x))} \qquad -\langle -\boldsymbol{n} | \boldsymbol{d} \rangle = -\mathrm{e}^{\mathrm{i}\frac{a}{2}(-\tau_x^g(\boldsymbol{x}_i) - \tau_y^g(\boldsymbol{x}_i - \boldsymbol{e}_x))}. \tag{20}$$

Thus, when using these relations, we find for $\boldsymbol{D}_x \cdot \boldsymbol{n}$ the following expression:

$$4as\langle \boldsymbol{D}_x | \boldsymbol{n} \rangle = (\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c} + \boldsymbol{d}) \cdot \boldsymbol{n} = \mathrm{i}2a\tau_x^g \tag{21}$$

in agreement with expression (16) for the connection. Note also that this parallel transport gives no phase term for $\boldsymbol{n} \cdot \boldsymbol{n}$:

$$\boldsymbol{n} \cdot (\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{b} - \boldsymbol{d}) = 4 + \mathrm{O}(a^2). \tag{22}$$

5. The angular momentum

Using the definition of l, we can show that the sum on all the *plaquettes* of $\boldsymbol{l}$ is invariant. It commutes with the Heisenberg Hamiltonian: $\{H, \boldsymbol{l}\} = 0$. Note also that in dimension $2+1$, the second relation of (6) no longer holds: $\boldsymbol{n}$ and $\boldsymbol{l}$ are no longer orthogonal. Let $h = \boldsymbol{n} \cdot \boldsymbol{l}$, the residual magnetization along $\boldsymbol{n}$. We can show that it is an invariant $\{H, h\} = 0$. Thus, in the continuum limit, the sum of $\boldsymbol{n} \cdot \boldsymbol{l}$ is also a conserved quantity [18].

In order to find an expression for h, we shall again use again the Liouville theorem. When computing $\mathrm{i}\{H_x, \boldsymbol{n}\}$ with the terms in the Hamiltonian involving the links in the $\boldsymbol{e}_x$ direction (that is when considering two spin chains in this direction) we then find

$$\frac{\partial \boldsymbol{n}}{\partial t} = \frac{cg^2}{2} \boldsymbol{l} \times \boldsymbol{n} + \frac{cg^2}{8} \boldsymbol{n} \times (2s\partial_x \partial_y \boldsymbol{n} + \partial_x \boldsymbol{\Gamma}_y). \tag{23}$$

We have set here $cg^2 = 8Ja^2$. The last four terms involving the *plaquette* (x, y) links in the $\boldsymbol{e}_y$ direction. Thus in order to keep the same notation as previously, we now have two spin chains where $\boldsymbol{a} \to \boldsymbol{b}$, $\boldsymbol{b} \to \boldsymbol{c}$, $\boldsymbol{c} \to \boldsymbol{d}$, $\boldsymbol{d} \to \boldsymbol{a}$. Therefore, $\boldsymbol{n} \to -\boldsymbol{n}$ and $\boldsymbol{D}_y \to -\boldsymbol{D}_x$. As $\boldsymbol{A}(-\boldsymbol{n}) = \boldsymbol{A}(\boldsymbol{n})$, we see that under this transformation, $\boldsymbol{\Gamma}_x = \boldsymbol{A}(-\boldsymbol{n}) \cdot \partial_x \boldsymbol{n}(x)\boldsymbol{n} \to \boldsymbol{\Gamma}_y$. Thus $\boldsymbol{\Gamma}_\mu$ behaves like $\boldsymbol{l}$. We then find in the $\boldsymbol{e}_y$ direction

$$\frac{\partial - \boldsymbol{n}}{\partial t} = \frac{cg^2}{2}\boldsymbol{l} \times -\boldsymbol{n} + \frac{cg^2}{8}(-\boldsymbol{n}) \times (2s\partial_y\partial_x((-\boldsymbol{n}) + \partial_y \boldsymbol{\Gamma}_x). \tag{24}$$

Thus globally, when we take into account all links, we find

$$\frac{\partial \boldsymbol{n}}{\partial t} = cg^2 \boldsymbol{l} \times \boldsymbol{n} + \tfrac{1}{8}\boldsymbol{n} \times (\partial_x \boldsymbol{\Gamma}_y - \partial_y \boldsymbol{\Gamma}_x). \tag{25}$$

In order to express h, the residual magnetization's component along $\boldsymbol{n}$, we can use the same kind of calculation which led us to equation (21). Indeed, h corresponds to the rotation of the frame $(\boldsymbol{u}, \boldsymbol{v})$ around $\boldsymbol{n}$. In the coherent state representation, this rotation is associated with the phase that appears in the definition of the scalar product (18). Using (19) and (20) we find the following expression for $\boldsymbol{n} \cdot \boldsymbol{l}$:

$$\begin{aligned} &4a^2 h = 4a^2 \langle \boldsymbol{n} | \boldsymbol{l} \rangle = \mathrm{i}\frac{a^2}{2}(\partial_x \tau_y^g - \partial_y \tau_x^g) \\ &h = \tfrac{1}{8}(\partial_x \boldsymbol{\Gamma}_y - \partial_y \boldsymbol{\Gamma}_x) \cdot \boldsymbol{n} = \tfrac{1}{8}(\partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n}).\boldsymbol{n} = \tfrac{1}{8} J^1. \end{aligned} \tag{26}$$

h is nothing but the density of the Pontryagin index [19]. It describes the chiral fluctuations of the staggered magnetization [15]. When integrated over the whole plane (x, y), it is indeed a conserved quantity with respect to the time evolution. These fluctuations are responsible for the angular momentum's component along $\boldsymbol{n}$. Therefore, using (25) and (26), we can conclude that the full expression of $\boldsymbol{l}$ is

$$\boldsymbol{l} = \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{cg^2 \partial t} - \frac{1}{8}(\partial_x \boldsymbol{\Gamma}_y - \partial_y \boldsymbol{\Gamma}_x) = \boldsymbol{L} + \boldsymbol{S}. \tag{27}$$

The first part of this expression, $\boldsymbol{L}$ is the angular momentum density associated with the observables degrees of freedom (the order parameter $\boldsymbol{n}$), also called the orbital part. The second part represents the generator of rotations associated with the inner degree of freedom, or classical spin, denoted $\boldsymbol{S}$ [20]. It corresponds to the free orientation of a plane orthogonal to $\boldsymbol{n}$, or local Berry's phase.

6. Expansion of the Hamiltonian density

When considering the inner links of a *plaquette*, the exact expression of the Hamiltonian density as a function of the new variables has the following form:

$$\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{c} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{c} = 4(a^4 \boldsymbol{l}^2 - s^2 \boldsymbol{n}^2) = 4a^4 \left(2\boldsymbol{l}^2 + \frac{s^2}{a^2}(\boldsymbol{D}_x^2 + \boldsymbol{D}_y^2) \right) - 4s^2. \tag{28}$$

We shall now consider the links between neighbouring *plaquettes*, and expand the Hamiltonian density, keeping terms of order less than or equal to a^4. We look first at the links in the $\boldsymbol{e}_x$ direction, that is for two spin chains in this direction.

$$\begin{aligned} \frac{1}{2s^2}[(\mathbf{1} \cdot \boldsymbol{a} + \boldsymbol{d} \cdot \mathbf{6}) + (\mathbf{5}\boldsymbol{c} + \boldsymbol{b} \cdot \mathbf{2})] = \frac{2a^4}{s^2}&\boldsymbol{l}^2 - \boldsymbol{n} \cdot [\boldsymbol{n}(x - 2a, y) + \boldsymbol{n}(x + 2a, y)] \\ &+ a\boldsymbol{D}_x \cdot (\boldsymbol{n}(x - 2a, y) - \boldsymbol{n}(x + 2a, y)) + a\boldsymbol{n} \cdot (\boldsymbol{D}_x(x + 2a, y) \\ &- \boldsymbol{D}_x(x - 2a, y)) + a^2 \boldsymbol{D}_x \cdot [\boldsymbol{D}_x(x + 2a, y) + \boldsymbol{D}_x(x - 2a, y)] \end{aligned}$$

$$
\begin{aligned}
&-a^2\boldsymbol{D}_y\cdot[\boldsymbol{D}_y(x+2a,y)+\boldsymbol{D}_y(x-2a,y)]+\frac{a^3}{s}\boldsymbol{l}\cdot(\boldsymbol{D}_y(x-2a,y)\\
&-\boldsymbol{D}_y(x+2a,y))+\frac{a^3}{s}\boldsymbol{D}_y\cdot(\boldsymbol{l}(x+2a,y)-\boldsymbol{l}(x-2a,y))\\
=&\frac{4a^4}{s^2}\boldsymbol{l}^2+4a^2(\partial_x\boldsymbol{n})^2-8a^2\boldsymbol{D}_x\cdot\partial_x\boldsymbol{n}+4a^2(\boldsymbol{D}_x)^2+8\frac{a^4}{s}\boldsymbol{l}\cdot\partial_x\boldsymbol{D}_y.
\end{aligned}
$$

We note that the terms which could, at the end of the expansion, contribute to the Pontryagin index ($a^3\boldsymbol{l}\cdot\partial\boldsymbol{n}$), cancel each other exactly. This result stems from the fact that the Pontryagin index is a topological invariant in dimension $D=2$ (or $1+1$). It is no longer relevant in dimension $2+1$.

This can be explained globally by noting that two neighbouring one-dimensional chains, of axis $\boldsymbol{e}_y$, and with indices 'i' and '$i+1$', will have, in a one-dimensional treatment, homotopy indices equal in absolute value (because the Pontryagin index, when considered as a function of x_i, is continuous and has integer value).

Owing to the local Néel ordering, the order parameters $\boldsymbol{n}_{i+1}$ are opposite to their vis-à-vis, $\boldsymbol{n}_i$. As a consequence, the homotopy indices will have opposite signs. Therefore globally, their summation along the axis $\boldsymbol{e}_x$ will not contribute to the total action [8]. We have shown here that the absence of such a contribution is even true locally, when considering only one *plaquette*.

The last four terms involving the *plaquette* (x,y) are the four links in the $\boldsymbol{e}_y$ direction. Thus in order to keep the same notation as previously, we now have two spin chains where $\boldsymbol{a}\to\boldsymbol{b}$, $\boldsymbol{b}\to\boldsymbol{c}$, $\boldsymbol{c}\to\boldsymbol{d}$, $\boldsymbol{d}\to\boldsymbol{a}$. Therefore, $\boldsymbol{n}\to-\boldsymbol{n}$ and $\boldsymbol{D}_y\to-\boldsymbol{D}_x$. Thus we find in the $\boldsymbol{e}_y$ direction

$$
\begin{aligned}
\frac{1}{2s^2}[(\boldsymbol{8}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{3})+(\boldsymbol{4}\cdot\boldsymbol{c}+\boldsymbol{7}\cdot\boldsymbol{d})]=&\frac{4a^4}{s^2}\boldsymbol{l}^2+4a^2(\partial_y\boldsymbol{n})^2\\
&-8a^2\boldsymbol{D}_y\cdot\partial_y\boldsymbol{n}+4a^2(\boldsymbol{D}_y)^2-8\frac{a^4}{s}\boldsymbol{l}\cdot\partial_y\boldsymbol{D}_x.
\end{aligned}
$$

Finally, we replace the auxiliary fields $2\boldsymbol{D}_\mu$ with $\partial_\mu\boldsymbol{n}+\frac{1}{s}\boldsymbol{\Gamma}_\mu$, and we find that, at lowest order, the Hamiltonian density has the following form

$$
H=2Ja^2\sum_{\langle i,j\rangle}[8a^2(\boldsymbol{l}+\tfrac{1}{8}(\partial_x\boldsymbol{\Gamma}_y-\partial_y\boldsymbol{\Gamma}_x))^2+s^2(\boldsymbol{\nabla}\boldsymbol{n})^2].
$$

Thus, in the Hamiltonian formulation, only the orbital part remains. We take the continuum limit in the above expression for H, that is $4a^2\sum_{\langle i,j\rangle}\longrightarrow\iint \mathrm{d}x\,\mathrm{d}y$, and replace $\boldsymbol{l}$ calculated from the Liouville theorem. The Hamiltonian for the low-energy excitations in a two-dimensional Heisenberg antiferromagnet is the nonlinear σ model:

$$
H=\frac{c}{2}\iint \mathrm{d}x\,\mathrm{d}y\,[g^{-2}(\boldsymbol{\nabla}\boldsymbol{n})^2+\frac{1}{(cg)^2}(\partial_t\boldsymbol{n})] \tag{29}
$$

where $c=2\sqrt{2}Jas$ and $g^2=\frac{2\sqrt{2}a}{s}$ which are the real physical observables. They now correspond to the values found using the renormalization group techniques [21].

7. Conjugate momenta

Now, we shall compute the term $\partial_x\boldsymbol{\Gamma}_y-\partial_y\boldsymbol{\Gamma}_x$, which is associated with the classical spin $\boldsymbol{S}$ of the order parameter. We shall use the space-curve formalism, where the derivatives of the field $\boldsymbol{n}$ are expressed through the Darboux-Ribaucour equations [3, 16]:

$$\partial_i \boldsymbol{n} = \kappa_g^i \boldsymbol{u} + \kappa_n^i \boldsymbol{v}$$
$$\partial_i \boldsymbol{u} = -\kappa_g^i \boldsymbol{n} + \tau_g^i \boldsymbol{v}$$
$$\partial_i \boldsymbol{v} = -\kappa_n^i \boldsymbol{n} - \tau_g^i \boldsymbol{u}$$

where $(\boldsymbol{u}, \boldsymbol{v})$ span the plane perpendicular to $\boldsymbol{n}$. Thus we can write:

$$\begin{aligned}\boldsymbol{S} = \partial_x \boldsymbol{\Gamma}_y - \partial_y \boldsymbol{\Gamma}_x = \partial_x(\tau_y \boldsymbol{n}) - \partial_y(\tau_x \boldsymbol{n})\\ = (\partial_x \tau_y - \partial_y \tau_x)\boldsymbol{n} + (\tau_g^x \kappa_g^y - \tau_g^y \kappa_g^x)\boldsymbol{u} + (\tau_g^x \kappa_n^y - \tau_g^y \kappa_n^x)\boldsymbol{v}.\end{aligned} \tag{30}$$

The extrinsic curvatures can be expressed as functions of the Euler angles:

$$\kappa_g^i = \sin\theta \sin\psi\, \partial_i \varphi + \partial_i \theta \cos\psi \qquad \text{and} \qquad \kappa_n^i = \sin\theta \cos\psi\, \partial_i \varphi - \partial_i \theta \sin\psi. \tag{31}$$

Together with (17), we can then write [22]

$$\boldsymbol{S} = \varepsilon^{ij} \sin\theta\, \partial_i \varphi \partial_j \psi \boldsymbol{v}' + \varepsilon^{ij} \partial_i \psi \partial_j \theta \boldsymbol{u}' + \varepsilon^{ij} \sin\theta\, \partial_i \theta \partial_j \varphi \boldsymbol{n} + \varepsilon^{ij} \cos\theta\, \partial_i \varphi \partial_j \theta \boldsymbol{u}' \tag{32}$$

where $\boldsymbol{u}' = \cos\psi \boldsymbol{u} - \sin\psi \boldsymbol{v} = \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\sin\theta \partial\varphi}$ and $\boldsymbol{v}' = \sin\psi \boldsymbol{u} + \cos\psi \boldsymbol{v} = \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial\theta}$. The $\varepsilon^{ij} \cos\theta \partial_i \varphi \partial_j \theta$ term does not have the same symmetry as the Pontryagin index, $J^1(\theta, \varphi) = \varepsilon^{ij} \sin\theta \partial_i \varphi \partial_j \psi$. Indeed, when considering configurations with fixed boundary conditions, that is when $\theta \to [0, \pi]$, $\varphi \to [0, 2\pi]$ and $\psi \to [0, 2\pi]$, its contribution vanishes. If we make the change $\theta \to \pi - \theta$, $\varphi \to \varphi$ and $\psi \to \psi$ in order to map the southern hemisphere on the northern hemisphere, then:

$$\begin{aligned}\int \mathrm{d}x\, \mathrm{d}y\, \varepsilon^{ij} \cos\theta \partial_i \varphi \partial_j \theta &= \int_0^{2\pi} \int_0^{\pi} \operatorname{cotan}\theta\, J^1(\theta, \varphi)\, \mathrm{d}\theta\, \mathrm{d}\varphi\\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \operatorname{cotan}\theta\, J^1\, \mathrm{d}\theta\, \mathrm{d}\varphi + \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \operatorname{cotan}\theta\, J^1\, \mathrm{d}\theta\, \mathrm{d}\varphi = 0\end{aligned}$$

because $f(\theta) = \operatorname{cotan}\theta$ and $J^1(\theta, \varphi)$ are odd for $\theta \to \pi - \theta$. For the orbital part, we have:

$$\boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial t} = \partial_t \theta \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial \theta} + \sin\theta \partial_t \varphi \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\sin\theta \partial\varphi} = \sin\theta \partial_t \varphi \boldsymbol{u}' + \partial_t \theta \boldsymbol{v}'$$

and then, when we project the relation (27) on $\boldsymbol{u}'$ and $\boldsymbol{v}'$, we conclude that

$$\Pi_\theta = \partial_t \theta - \frac{\alpha}{8\pi} \varepsilon^{ij} \sin\theta \partial_i \varphi \partial_j \psi$$
$$\Pi_\varphi = \sin\theta \partial_t \varphi - \frac{\alpha}{8\pi} \varepsilon^{ij} \partial_i \psi \partial_j \theta$$

where the topological angle α is equal to $\alpha = \pi$. Therefore, in the continuum limit, the Heisenberg Hamiltonian in dimension $D = 2 + 1$ has the following form

$$\begin{aligned}H = \frac{c}{2} \iint \mathrm{d}x\, \mathrm{d}y \Bigg[g^{-2} (\boldsymbol{\nabla n})^2 + g^2 \left(\Pi_\theta - \sin\theta \frac{\alpha}{8\pi} \varepsilon^{ij} \partial_i \varphi \partial_j \psi \right)^2\\ + \frac{g^2}{\sin^2\theta} \left(\Pi_\varphi - \sin\theta \frac{\alpha}{8\pi} \varepsilon^{ij} \partial_i \theta \partial_j \psi \right)^2 \Bigg]\end{aligned} \tag{33}$$

where $c = 2\sqrt{2} Jas$ is the velocity of the magnons in two dimensions and $g^2 = \frac{2\sqrt{2}a}{s}$ the coupling constant of the nonlinear σ model. The Lagrangian density associated with this

Hamiltonian is the nonlinear σ model with a Hopf term:

$$\mathcal{L} = \frac{1}{2g^2}\iint \mathrm{d}x\,\mathrm{d}y\left[c^{-2}\left(\frac{\partial \boldsymbol{n}}{\partial t}\right) - (\boldsymbol{\nabla}\boldsymbol{n})^2\right] + \frac{\alpha}{8\pi}\iint \mathrm{d}x\,\mathrm{d}y\,\varepsilon^{ijk}\sin\theta\,\partial_i\varphi\partial_j\theta\partial_k\psi$$
$$= \frac{1}{2g^2}\iint \mathrm{d}x\,\mathrm{d}y\left[c^{-2}\left(\frac{\partial \boldsymbol{n}}{\partial t}\right) - (\boldsymbol{\nabla}\boldsymbol{n})^2\right] + \frac{\alpha}{8\pi}\iint \mathrm{d}x\,\mathrm{d}y\,\boldsymbol{J}\cdot\boldsymbol{A}. \tag{34}$$

$\alpha = \pi$ is the topological angle associated with the statistic of the excitations. $J^k = \varepsilon^{ijk}\varepsilon^{abc} n^a\partial_i n^b\partial_j n^c$ is the topological current constructed from the vector field $\boldsymbol{n}$ and $\boldsymbol{A}$ is the gauge fields from which it derives: $\mathrm{rot}\,\boldsymbol{A} = \boldsymbol{J}$.

8. Conclusion

In order to find the correct values of the physical observables c, the celerity of the magnons, and g^2, the strength of the coupling, it is crucial to take into account the role of the auxiliary fields in the expansion of the Hamiltonian density.

These fields assure the conservation of the number of degrees of freedom, when taking the continuum limit à la Affleck. They allow us to naturally introduce a gauge field, which for certain configurations of $\boldsymbol{n}$, cannot be gauged out. Moreover, in contrast to the $D = 1+1$ case, the angular momentum associated with the continuum order parameter of the Heisenberg model in dimension $D = 2+1$ is no longer perpendicular to the order parameter of the nonlinear σ model. Indeed it possesses a classical spin part. Owing to the fact that the true order parameter in $D = 2+1$ is now a full trihedra.

As a consequence, we have shown that this gauge field is responsible for the presence of a Hopf term in the Lagrangian action which could modify the statistics of the excitations, as suggested initially by Dzyaloshinskii *et al*.

References

[1] Haldane F D M 1983 *Phys. Rev. Lett.* **50** 1153
Haldane F D M 1983 *Phys. Lett.* **93A** 464
[2] Korepin V E and Faddeev L D 1975 *Teor. Mat. Fiz.* **25** 1039
Korepin V E and Faddeev L D 1978 *Phys. Rep.* C **42** 1
[3] Felsager B and Leinaas J M 1980 *Nucl. Phys.* B **166** 162–88
[4] Baez J C, Bishop A R and Dandoloff R 1991 *Mod. Phys. Lett.* B **5** 2003
[5] Dzyaloshinskii I, Polyakov A and Wiegmann P 1988 *Phys. Lett.* **127A** 112
[6] Haldane F D M 1988 *Phys. Rev. Lett.* **61** 1029
Dombre T and Read N 1988 *Phys. Rev.* B **38** 7181
Fradkin E and Stone M 1988 *Phys. Rev.* B **38** 7215
[7] Affleck I 1988 *Phys. Rev.* B **37** 5186
[8] Shankar R and Read N 1990 *Nucl. Phys.* B **336** 457–74
[9] Mikeska H-J 1980 *J. Phys. C: Solid State Phys.* **13** 2913
[10] Affleck I 1984 *Nucl. Phys.* B **257** 397–406
[11] Affleck I 1988 Field theory methods and quantum critical phenomena *Champs, Cordes et Phénomènes Critiques (Proc. Les Houches Session XLIX)* ed E Brézin and J Zinn-Justin (Amsterdam: Elsevier)
[12] 'tHooft G 1981 *Nucl. Phys.* B **190** 455
[13] Lieb E, Schultz T D and Mattis D C 1961 *Ann. Phys.* **16** 407
[14] Villain-Guillot S and Dandoloff R 1996 *Mod. Phys. Lett.* B **10** 1289
[15] Auerbach A 1994 *Interacting Electrons and Quantum Magnetism* (New York: Springer)
[16] Balakrishnan R, Bishop A R and Dandoloff R 1993 *Phys. Rev.* B **47** 5439
[17] Zeldovich Y A B, Ruzmaikin A A and Sokolov D D 1986 *Magnetic Fields in Astrophysics* (Paris: Mir) ch 6
Zeldovich Y A B, Ruzmaikin A A and Sokolov D D 1985 *Sov. Phys.–Dokl.* **30** 756
Kuznetsov E A and Mikhailov A V 1980 *Phys. Lett.* **77A** 347

[18] Ivanov B A and Kolezhuk A K 1995 *Phys. Rev. Lett.* **74** 1859
[19] Trimper S 1979 *Phys. Lett.* **70A** 114
[20] Felsager B 1981 *Geometry, Particles and Fields* (Odense: Odense University Press)
[21] Chakravarty S, Halperin B and Nelson D 1988 *Phys. Rev. Lett.* **60** 1057
[22] Percacci R and Pak N K 1991 *Phys. Rev.* D **43** 1375